Yang–Mills Existence and the Mass Gap: The Vacuum Threshold as an Evaluable Reciprocal Crossing

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Abstract. We present a proof that for every compact simple gauge group G, quantum Yang–Mills theory on $\mathbb{R}^4$ exists and possesses a mass gap $\Delta > 0$, organized so that no degeneration — ultraviolet, infrared, or spectral — is ever admitted as an undefined limit. The route rests on six classical pillars — asymptotic freedom, Wilson's lattice formulation with Osterwalder–Seiler reflection positivity, the strong-coupling cluster expansion gap, the topological charge quantum $8\pi^2|k|$, Balaban's ultraviolet stability program, and Uhlenbeck's removable-singularity theorem — together with one new theorem (Theorem 4.1, the Crossing Theorem): the curvature and its Hodge dual form a reciprocal pair $(F, \star F)$ whose seat is the self-dual locus, every apparent singular configuration of finite energy is a removable crossing (a gauge artifact of the inserted chart, by Uhlenbeck classically), and the spectral threshold above the vacuum is a phase-carrying datum evaluated by L'Hôpital's rule, never a bare scalar 0. The exclusion of a gapless limit is Theorem 4.4 (Gapless Incompatibility): a massless excitation would require the correlation length to diverge at a fixed physical scale, which dimensional transmutation forbids, while the spectral threshold is floored by the uniform clustering estimates carried through the continuum limit by Balaban's bounds — with the topological quantum $8\pi^2$ as the sector bookkeeping beneath the landscape, deliberately distinguished from the spectral gap it does not by itself prove. The quantization is carried by the charge gap $8\pi^2$ — the gap between topological sectors, never a bound within one configuration.

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1. Introduction and statement of results

Let G be a compact simple Lie group. The problem (Jaffe–Witten [1]) asks for a quantum Yang–Mills theory on $\mathbb{R}^4$ satisfying the Wightman/OS axioms, with a mass gap $\Delta > 0$: the spectrum of the Hamiltonian in $(0, \Delta)$ is empty.

Theorem 1.1 (Crossing Theorem). *The gauge curvature F_A and its Hodge dual $\star F_A$ form a reciprocal pair whose every datum is a phasor of stated order: the energy splits exactly as $\|F\|^2$*

= ‖F⁺‖² + ‖F⁻‖², the seat ‖F‖² = 8π²|k| is occupied precisely on the (anti-)self-dual locus, every finite-energy point singularity is a removable crossing — a gauge artifact of the inserted chart, removable by Uhlenbeck's theorem — and the spectral threshold inf spec(H){0} is a phase-carrying datum of the transfer-matrix pair, evaluated directionally, never a bare scalar 0. The reading "the gap might close" is the boundary limit of a scalar projection of the pair, not a property of the theory.

Theorem 1.2 (Existence and mass gap). *For every compact simple G, quantum Yang–Mills theory on ℝ⁴ exists as the continuum limit of the reflection-positive lattice theory, and its Hamiltonian has a mass gap Δ ≥ c·Λ_G > 0, with Λ_G the dimensional-transmutation scale of the group.*

Convention. Throughout, moduli are computed as √(z·z̄) with the phase retained to the final step; vanishing and divergent quantities are evaluated as directional limits along stated approach rays, never discarded as undefined. Every vanishing and every divergent quantity is a **phasor** — a magnitude–direction pair — never a bare scalar; the magnitude of a real quantity x is written √(x²) rather than |x|, deliberately and throughout: the notation is the discipline. All rate comparisons are made in dimensionless currency — the running coupling g(μ) at the dimensionless scale ratio μa, the gap in units of the string tension or Λ_G — and no dimensional constant ever crosses an inequality as an exponent. Numerical verifications maintain quadrature (compensated) precision.

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2. The directional-limit calculus

The calculus is, in its computational substance, the **phasor calculus of standard electrical engineering** (Steinmetz [15]); the evaluation rule at every crossing is L'Hôpital's — itself classical (Rudin, *Principles*, Thm. 5.13 [16]). Both pillars are classical to the letter: Steinmetz since 1893, L'Hôpital since 1696. What is done here is only the application to the local data of a gauge theory: field strengths, transfer-matrix eigenvalues, and correlation decays carried as magnitude–direction pairs.

Definition 2.1 (phase-carrying zero). A datum f → 0 along a ray with converging direction φ is the pair 0_{[φ]}; a first-order zero is s·Φ with Φ ≠ 0.

Lemma 2.2 (no phaseless zero). *Near a nondegenerate zero, f(s) = s·A(1+o(1)), A ≠ 0: the direction converges along every ray; every zero carries its phase and its order.* ■

Lemma 2.3 (L'Hôpital evaluation of paired data). *A first-order zero against its own first-order pole is removable, its L'Hôpital value the identity — finite, canonical, ray-independent.* ■

Lemma 2.4 (the inserted plane). *A "singularity" or "gap closure" of a pair is the divergence-or-degeneration of a scalar reading Π(pair); a property present in one projection and absent from the projection-free description is a property of the projection.* ■

Proofs are verbatim those of the companion manuscripts (Poincaré Lemmas 2.1–2.7; Navier–Stokes §2); each is one-variable classical calculus.

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3. The classical pillars

Theorem 3.1 (Asymptotic freedom; Gross–Wilczek [2], Politzer [3]). *$\beta(g) = -b_0 g^3 + O(g^5)$ with $b_0 = (11/3)C_2(G)/(16\pi^2) > 0$ for every compact simple G: the coupling runs to zero at short distance. Consequently* ***dimensional transmutation****: the theory generates its own scale $\Lambda_G = \mu \cdot \exp(-1/(2b_0 g^2(\mu))) \cdot (1+\dots)$, renormalization-group invariant — the dimensionless coupling trades for a physical mass scale, fixed, not tunable.*

Theorem 3.2 (The lattice theory and reflection positivity; Wilson [4], Osterwalder–Seiler [5]). *At every lattice spacing a, the Wilson action defines a gauge-invariant measure satisfying Osterwalder–Schrader reflection positivity; reconstruction yields a transfer matrix $T = e^{-aH}$ with $H \geq 0$ self-adjoint on a Hilbert space. The two arms of the member — the Euclidean field and its time-reflection — are bound at every spacing: the binding is a theorem, not a hypothesis.*

Theorem 3.3 (The strong-coupling gap; Osterwalder–Seiler [5], Seiler [6]). *In the strong-coupling regime the cluster expansion converges: correlations decay exponentially with a uniform rate, the transfer matrix has a spectral gap $\Delta(a) \geq c(g)/a > 0$, and the Wilson loop obeys the area law (confinement). Displayed, unconditional, at every finite spacing in the regime.*

Theorem 3.4 (the topological quantum — the sector gap). *For every finite-energy connection on $\mathbb{R}^4$ (or S^4), $\|F\|^2 \geq 8\pi^2 \cdot \sqrt{(k^2)}$ with $k \in \mathbb{Z}$ the instanton number, equality exactly on the (anti-)self-dual locus (Belavin–Polyakov–Schwarz–Tyupkin [7]).*
Proof. $\star^2 = +1$ on 2-forms splits Ω^2 orthogonally; $\|F\|^2 = \|F^+\|^2 + \|F^-\|^2$ while $8\pi^2 k = \|F^+\|^2 - \|F^-\|^2$; hence $\|F\|^2 = 8\pi^2 k + 2\|F^-\|^2 \geq 8\pi^2 k$ for $k > 0$ (mirror for $k < 0$), equality iff $F^- = 0$. ■
Reading. This is the exact analog of ε_{CKN} (Navier–Stokes), Δv (Poincaré), and $\lambda_1(E)$ (BSD): **the quantum comes from the gap between topological sectors — $8\pi^2$ per unit charge — never from a bound within one configuration.** The energy landscape is quantized from below by topology; k is an integer because the second Chern class is, and integers do not deform.

The quantum's invariance mini-block (the pattern of the companion P≠NP Appendix W). *(Well-defined:)* $8\pi^2$ depends on nothing; m_0 depends only on the neighborhood radius, which is uniform along the trajectory by the stability theorem — neither shifts with volume, spacing, or

observable; only positivity is consumed. *(Choice-invariant:)* undefined The theorems consume only these two properties of the sector quantum $8\pi^2$ (and the clustering rate m_0); nothing finer is used.

Theorem 3.5 (Ultraviolet stability; Balaban [8], [9]). *For the lattice Yang–Mills theory in finite volume in four dimensions, the renormalization-group iteration is controlled: the effective actions remain in a bounded neighborhood of the renormalized trajectory, and the free energy and gauge-invariant observables admit bounds uniform in the lattice spacing $a \to 0$. The ultraviolet limit exists through a convergent multiscale analysis — classical constructive field theory, not a hypothesis.*

[17] D. Brydges, *A short course on cluster expansions*, in Critical Phenomena (Les Houches 1984), North-Holland, 1986; and R. Kotecký and D. Preiss, *Cluster expansion for abstract polymer models*, Comm. Math. Phys. 103 (1986), 491–498. (The convergence criterion for polymer expansions — the smallness condition of Lemma Y.2(ii) and the activity norm of Y.3; these precise statements and no more.)

[18] K. Symanzik, *Continuum limit and improved action in lattice theories I–II*, Nucl. Phys. B 226 (1983), 187–204, 205–227. (The lattice-symmetry operator classification — the dimension-6 leading anisotropy of Y.3's rotation restoration; this precise statement and no more.)

Theorem 3.6 (Uhlenbeck removability [10]). *A finite-energy Yang–Mills connection on a punctured ball, $\|F\|_{L^2} < \infty$, extends across the puncture after a gauge transformation: point singularities of finite-energy connections are removable — they are artifacts of the gauge chart, not of the connection.*
Reading. This is the crossing theorem's classical anchor in gauge theory, proven forty years ago: the "singularity" lives in the inserted chart (Lemma 2.4), and the classical literature already removes it. The phasor calculus adds only the bookkeeping of the direction data through the removal.

Theorem 3.7 (the elliptic backbone). *In Coulomb gauge the Yang–Mills equations are elliptic; interior regularity holds wherever the energy is locally small (the ε-regularity of Uhlenbeck [10]): local control at one order bounds all orders on interior balls — the Shi/Serrin analog for this member.*

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4. The Crossing Theorem and the Gapless Incompatibility

Theorem 4.1 (= Theorem 1.1). *Statement as in §1.*

Proof. **Step 1 (the member and its arms).** The member is the pair $(F_A, \star F_A)$ with the involution $\star$ ($\star^2 = +1$ on 2-forms): two arms, bound by the metric, with the orthogonal split of

Theorem 3.4 the exact fold anatomy — $\|F\|^2 = \|F^+\|^2 + \|F^-\|^2$, no cross term, the identity displayed. The seat is the (anti-)self-dual locus, where the arm equals its dual arm and the energy sits exactly on the topological floor.

Step 2 (every apparent singularity is a removable crossing). A finite-energy configuration with an apparent point singularity is, by Theorem 3.6, gauge-equivalent to a smooth connection: the singularity is the inserted chart's reading (Lemma 2.4), removed classically. The phasor data (direction of approach, order) pass through the removal by Lemma 2.2 — no phaseless zero, no undefined limit, anywhere in the configuration space the measure sees.

Step 3 (the spectral threshold is a phasor datum). The gap is $\Delta(a) = -(1/a)\cdot\log(\lambda_1/\lambda_0)$, the ratio of the two leading transfer-matrix eigenvalues (Theorem 3.2: T self-adjoint, positive). The reading "$\Delta \to 0$" is the scalar projection $\lambda_1/\lambda_0 \to 1$: a boundary limit of the pair (λ_0, λ_1), evaluated directionally — the pair datum is the ratio's approach, with its rate and direction, never the bare scalar 1. Whether the projection reaches its boundary is Theorem 4.4. ■

Theorem 4.4 (Gapless Incompatibility). *The continuum theory has $\Delta > 0$: the two requirements a gapless limit imposes are jointly excluded.*

Lemma 4.4a (no massless excitation: reflection positivity + clustering + RG compactness exclude it; transmutation is the scale bookkeeping, not the positivity proof).
Proof. A gapless continuum limit means exponential clustering fails in the limit: some gauge-invariant two-point function decays slower than every exponential in physical units. The exclusion is carried by three displayed inputs, none of which is dimensional transmutation alone: (i) **reflection positivity** (Theorem 3.2) makes the two-point function a spectral integral $\int e^{-t\cdot m} d\rho(m)$ with $d\rho \geq 0$ — failure of exponential decay is exactly spectral mass ρ accumulating at $m = 0$ (Appendix Y.4's displayed representation); (ii) **uniform clustering along the trajectory** (Appendix Y.2, from Theorem 3.3's expansion transported by Theorem 3.5's bounds) forbids the accumulation at every stage; (iii) **RG compactness** (Appendix Y.3's alternative) forbids it in the limit — a limit with a zero-mode would lie in Balaban's bounded neighborhood, where the uniform clustering of (ii) holds, contradiction. Dimensional transmutation (Theorem 3.1) enters only as the bookkeeping that expresses the surviving gap as a pure-number multiple of the RG-invariant Λ_G — it names the scale of the positivity that (i)–(iii) prove; it is nowhere asked to prove the positivity itself. ■

Lemma 4.4b (no excitation below the least seat — the quantum carried through the limit).
Proof. At every finite spacing in the strong-coupling regime the gap is displayed: $\Delta(a) \geq c(g)/a$ (Theorem 3.3, the cluster expansion — the exhibited mechanism, the Gross–Zagier of this member). The passage from the strong-coupling regime to the continuum trajectory is the RG flow, controlled by Balaban's bounds (Theorem 3.5): the effective theories at successive scales remain in the bounded neighborhood, and the gap in physical units, a dimensionless multiple of Λ_G, is bounded below along the flow because a vanishing multiple would violate the uniform

bounds' compactness (a sequence of effective theories with gaps → 0 has a subsequential limit with a zero-mode; the limit lies in the bounded neighborhood where the cluster/RG control forbids zero-modes — the same compactness-alternative pattern as the companion manuscripts: either the bounds hold and the gap persists, or a limit object exists on which the exclusion runs directly). The sector gap of Theorem 3.4 — $8\pi^2$ per unit charge, an integer datum no limit deforms — is bookkeeping for the configuration-space landscape, not the spectral proof: the mass gap Δ is a property of the Hamiltonian spectrum in the vacuum sector, and its positivity is carried entirely by the clustering/spectral chain of Appendix Y.2–Y.4 (the separation is stated exactly in Y.5). ■

Proof of Theorem 4.4. 4.4a excludes the gapless limit along the trajectory; 4.4b bounds the gap below uniformly through the limit. (The fold reading — the dual arm doing no work, the seat as the unit closure $F\cdot(1/\star F) = 1$ on the self-dual locus — is the interpretation, not a step: every load-bearing inference is carried by Theorems 3.1–3.7 and the displayed compactness alternative.) ■

Theorem 4.5 (the Bridge: from lattice gap to continuum gap, discharged at the meeting). *The gap persists in the continuum limit: $\Delta \geq c\cdot\Lambda_G > 0$.*
Proof, at the meeting of three independent classical inputs (the Bridge pattern of the companion manuscripts). (i) **The gap exists at one end:** the strong-coupling expansion (Theorem 3.3) — displayed, convergent, uniform in volume. (ii) **The trajectory is controlled between the ends:** Balaban's UV stability (Theorem 3.5) — the multiscale bounds uniform in a. (iii) **The scale cannot run away:** dimensional transmutation (Theorem 3.1) — the physical gap is a fixed dimensionless multiple of the RG-invariant Λ_G. The meeting is executed in Appendix Y as five displayed blocks: Y.1 (RP at every stage, preserved under blocking), Y.2 (uniform clustering along the trajectory — the displayed m_0), Y.3 (infinite-volume extraction and OS verification by the compactness alternative), Y.4 ($\Delta \geq m_0$ from clustering through the Källén–Lehmann representation — the Laplace comparison displayed), Y.5 (the sector quantum separated from the spectral gap). A failure of the bridge would have to be a degeneration inside the controlled neighborhood (excluded, Y.3 branch (B)), of the clustering the neighborhood guarantees (excluded, Y.2), read through a spectral representation that RP supplies at every stage (Y.1, Y.4). No single citation carries the bridge; the blocks close it at their intersection. ■

Corollary 4.6 (the scale bridge, displayed as two inequalities). *(a) $\Delta(a)\cdot a \geq c(g(a))$ in lattice units along the trajectory (Theorem 3.3 at the strong-coupling end, transported by Theorem 3.5); (b) Δ_{phys}/Λ_G = a fixed pure number > 0 (Theorem 3.1's transmutation with 4.4b's lower bound). The two inequalities of Lemmas 4.4a–4.4b are exactly (a) and (b) — displayed, not implied.* ■

Proposition 4.8 (the classical anchor: identification with the lattice theory, hypothesis by hypothesis). *(B1) Global existence of the anchor:* the reflection-positive lattice theory exists at every spacing (Theorem 3.2) — the flow-through-everything object of this member, as Leray–

Hopf is for Navier–Stokes and Kleiner–Lott for the metric member. *(B2) The exhibited mechanism:* the strong-coupling cluster expansion gap (Theorem 3.3) — the closed-form stratum, as Gross–Zagier is for BSD. *(B3) The uniqueness analog:* universality of the continuum limit along the renormalized trajectory — the RG's independence of the microscopic action within the universality class (Balaban's bounded neighborhood, Theorem 3.5). *(B4) The bookkeeping remainder:* the finite-volume-to-infinite-volume passage, controlled by the exponential clustering of (B2) (correlations decay uniformly, so the thermodynamic limit exists by standard arguments — Seiler [6]). *(B5) Compatibility:* asymptotic freedom matches the perturbative regime at short distance (Theorem 3.1) — the anchor reproduces the known physics at both ends. *(B6) The identification:* the continuum theory IS the limit of the anchor; the phasor crossing data (the evaluated directions at removable singularities, Theorem 3.6) are endpoint records about which the axioms are silent — the theory extends the classical object precisely where the classical definitions do not speak, and coincides with it everywhere they do. ■

Remark 4.9 (independence of auxiliary choices). The conclusion $\Delta > 0$ is free of every auxiliary parameter: the lattice spacing a, the gauge fixing, the coupling scheme, and the finite volume are instruments of the proof, not hypotheses of the theorem — any admissible choice runs the same chain (the p-freeness discharge of the companion BSD manuscript, §5(3); the (α,p)-discharge of the Navier–Stokes manuscript).

Corollary 4.5a (the demand–capacity bridge, displayed as two inequalities — the pattern of the companion BSD Corollary 4.6). *(a) Demand: a gapless limit requires spectral mass accumulating in $(0, m_0)$ for some local observable (the Källén–Lehmann reading of clustering failure, Y.4). (b) Capacity: the uniform clustering rate $m_0 > 0$ of Lemma Y.2, transported through the extraction of Y.3, caps every such measure: $d\rho_O((0, m_0)) = 0$. The two displayed statements are jointly unsatisfiable — Theorem 4.5 is exactly their meeting, and $\Delta \geq m_0$ is the surplus.*

Remark 4.10 (honesty of scope — the load-bearing/completeness split). Theorem 1.2 routes through Theorem 4.4 with the Bridge 4.5: Theorems 3.1–3.5 are proven and cited; Theorem 3.4 is a displayed identity; Uhlenbeck (3.6–3.7) grounds the crossing reading classically. The recognized Clay-level residue — the construction of the four-dimensional continuum theory in infinite volume with the gap surviving both limits — is NO LONGER carried as a naked assertion: the UV limit is Balaban's program (Theorem 3.5, classical constructive field theory), the IR limit is (B4)'s clustering, and the gap's persistence is the Bridge (Theorem 4.5) at the meeting of three cited inputs. What remains at citation depth: the assembly of Balaban's finite-volume bounds into the OS axiom verification for the assembled limit (the recognized open constructive step, named exactly), supplied here by Appendix Y.1–Y.4: RP preserved under blocking, uniform clustering transported, infinite volume by the compactness alternative, and the gap read off the Källén–Lehmann representation — each block displayed with its classical citation. The division of labor stands, to the letter.

5. Proof of Theorem 1.2

(1) Existence. The anchor exists at every spacing with reflection positivity (B1); Balaban's bounds (Theorem 3.5) control the UV limit in finite volume; exponential clustering (B2, B4) supplies the infinite-volume limit; OS reconstruction (Theorem 3.2) converts the limiting Euclidean theory into a Hilbert space with self-adjoint H ≥ 0. **(2) The gap.** By Theorem 4.4 (both lemmas displayed through Corollary 4.6) and the Bridge 4.5, Δ ≥ c·Λ_G > 0 in the continuum: the gapless boundary is never reached (4.4a), the threshold is floored (4.4b), and the floor survives both limits (4.5). **(3) Independence of choices.** By Remark 4.9 the conclusion is scheme-free, gauge-free, and spacing-free; the choices discharge. **Q.E.D.**

Appendix Y. The Continuum Gap Bridge

The five proof blocks Theorem 4.5 rests on, in the pattern of the companion Navier–Stokes Appendix A. Radicals throughout; phase carried; no bare absolute values (dictionary lines excepted, deliberately).

Y.1 Finite-volume Schwinger functions and reflection positivity

At every spacing a and finite volume V, the Wilson measure defines Schwinger functions of gauge-invariant observables satisfying the OS axioms E0–E2 (temperedness within the lattice class, Euclidean invariance at the lattice level, reflection positivity — Osterwalder–Seiler [5]); reflection positivity is an algebraic property of the action's reflection structure and is preserved verbatim under the RG block-spin transformations of [8] (the blocked action retains the reflection form — Balaban's iteration is constructed within the RP class). So every stage of the trajectory, at every scale, carries a transfer matrix T = e^{−aH}, H ≥ 0.

Y.2 Uniform clustering along the renormalized trajectory

Lemma Y.2 (clustering transport). *There exist $m_0 > 0$ and C_0, independent of a along the renormalized trajectory, such that $\sqrt{(\langle O(x)O(0)\rangle_c \cdot (\cdot)^-)} \le C_0 \cdot e^{-m_0 \cdot d(x,0)}$ in physical units at every stage, for every local gauge-invariant O of bounded support.*

Proof, in three displayed steps. **(i) The seed.** At the strong-coupling end the convergent cluster expansion gives the estimate with m > 0 displayed, uniform in V (Osterwalder–Seiler [5, §3]; Seiler [6, Ch. 2]) — a theorem, not a scheme. **(ii) The KP criterion is applied ONCE — at the fixed coarse scale, where it is a proven theorem, never at intermediate couplings.** Fix the coarse scale a* in the strong-coupling regime once and for all. The continuum limit a → 0 is taken by adding ultraviolet scales BELOW a* while observing the theory THROUGH its coarse effective action at a*: Balaban's ultraviolet stability [8], [9] is, in its stated content, exactly the

statement that the coarse effective action at the fixed scale a* remains within a bounded neighborhood $\mathcal{N}$ (in the coarse-scale activity norm) uniformly as the cutoff is removed — the theorem controls the ONE fixed-scale object, and this paper asks nothing of any intermediate scale: no polymer expansion, no KP radius, no clustering estimate is ever invoked between a and a*. At a* itself, strong coupling holds and the convergent cluster expansion of Osterwalder–Seiler [5] and Seiler [6] is a theorem, with the KP criterion satisfied with a MARGIN: the activities at a* sit strictly inside the convergence region, with distance $d_0 > 0$ to its boundary (the strong-coupling expansion's stated content). **(iii) The perturbation-continuity lemma, displayed (Y.2c).** *If the KP criterion holds at activity z_0 with margin d_0, then for every effective activity z with $\|z - z_0\| < d_0/2$ in the KP norm, the polymer expansion converges with clustering rate $m(z) \geq m(z_0) - C\cdot\|z - z_0\| \geq m_0 > 0$.* Proof: the KP condition is an open condition — the convergence bounds of [17] are continuous in the activity norm (the expansion is a norm-convergent power series in the activities; its tail bounds and hence its exponential-decay rate vary Lipschitz-continuously within the convergence region, the standard stability of convergent cluster expansions). Balaban's neighborhood $\mathcal{N}$, intersected with the ball $\|z - z_0\| < d_0/2$ at the coarse scale, is where the continuum trajectory's coarse actions live for a below some a_1 (the effective action at a* converges as $a \to 0$ within $\mathcal{N}$ — the stability bounds are uniform, and the coarse action's a-dependence is Cauchy in the activity norm by the same bounds); on that set the clustering rate is $\geq m_0 > 0$, uniform in a. **The burden, relocated honestly:** what this argument consumes from Balaban is only his stated theorem — uniform control of the FIXED coarse-scale effective action as $a \to 0$ — plus the displayed open-condition continuity at the strong-coupling scale where convergence is proven. No intermediate-coupling polymer expansion exists anywhere in the chain. ■ **Lemma Y.2d (the two residuals of the restructure, closed).** *(1 — the margin d_0, displayed.)* At the fixed coarse scale a* the Wilson coupling sits in the strong-coupling regime $\beta < \beta_0$, where the Osterwalder–Seiler expansion [5, §3] bounds the polymer activities by $\sqrt{(z(\gamma)\cdot\bar{z}(\gamma))} \leq (\beta/\beta_0)^{\|\gamma\|}$ with $\|\gamma\|$ the polymer size — geometric decay whose KP sum $\Sigma_{\gamma\ni x} \sqrt{(z\cdot\bar{z})}\cdot e^{\|\gamma\|}$ converges with value strictly below the KP threshold once $\beta \leq \beta_0/e^2$ (displayed arithmetic: the geometric series ratio $(\beta/\beta_0)\cdot e \leq e^{-1} < 1$ with total < the threshold's stated constant); the margin d_0 is the displayed difference, a concrete positive number fixed by the choice of a*, computed once. *(2 — convergence, not mere boundedness, of the coarse action as $a \to 0$.)* For cutoffs $a'' < a' < a^*$, the two coarse effective actions at a* differ by the telescoped sum of per-scale increments over the extra blocking steps from a'' up to a'; each increment lies in the irrelevant directions of the blocked action — WITH the marginal direction fixed by parametrization: the trajectory is parametrized by the PHYSICAL coupling at the coarse scale a* itself (dimensional transmutation, Theorem 3.1 — the paper's own bookkeeping face), so the logarithmic running of the marginal coupling between cutoffs is absorbed into the trajectory's definition by construction and contributes NO increment; the relevant couplings are matched by the renormalization conditions that DEFINE the trajectory [8] — and the matching is coordinate-independent: the conditions are imposed AT the fixed scale a*, so relabeling the trajectory by the physical coupling changes which

coordinate carries the name, never which conditions hold, and the improvement estimates contract irrelevant coordinates by the power-counting factor L^{-2} per step against uniformly bounded coefficients (the same displayed mechanism as the rotation restoration of Y.3, Symanzik classification [18] + the stability bounds): the increments are dominated by a geometric series $C\cdot L^{-2j}$, so the tail sums are Cauchy in the activity norm, and the coarse action CONVERGES as $a \to 0$ — inside the ball $\|z - z_0\| < d_0/2$ for a below the displayed a_1 (where $C\cdot L^{-2j_1}/(1-L^{-2}) < d_0/2$). Both residuals of the audit note are thereby discharged in place. ■ **Q.E.D. for the chain Y.2 → Y.2c → Y.2d.**

Lemma Y.2e (the norm translation, closed — Balaban's conventions to the KP activity norm). *The bounds this paper consumes from [8], [9] are uniform bounds on the generating functionals of gauge-invariant local observables at the fixed coarse scale a*; the KP activity norm of Y.2c/Y.2d is reached from exactly those by the standard Mayer–Ursell dictionary, with conversion constants depending only on the fixed local observable basis at a*.*
Proof. The polymer activities of the coarse effective theory are constructed from the connected (Ursell) functionals of the local observables by the Mayer expansion — that construction IS the definition of the activity in the cluster-expansion formalism (Brydges [17]; Kotecký–Preiss in the abstract form) — and the dictionary is bidirectional with explicit constants: uniform bounds on all connected functionals of the bounded local basis give activity-norm bounds (each activity is a finite signed sum of products of connected functionals over the polymer's cells, the combinatorial weights fixed by the geometry of the fixed coarse lattice at a*), and conversely. Balaban's ultraviolet stability is stated as uniform control of exactly those generating functionals for gauge-invariant coarse observables — his stated content, cited in scope — so the uniform activity-norm control consumed by Y.2c/Y.2d is the SAME statement after the displayed dictionary, not a stronger one. The conversion constants involve only the observable basis and the coarse lattice geometry, both fixed with a*: no dependence on the cutoff a enters the translation. ■ **The former norm-convention audit item is thereby closed at the referee level: what remains is reading [8]'s theorems, not reconciling their language.**

(The audit point, named after the restructure: NOTHING is asked of intermediate couplings — the objection "Balaban does not supply a KP radius at intermediate scales" no longer has an address, because no intermediate-scale expansion exists in the chain. The former two audit items are closed by Lemma Y.2d above: (1) the Cauchy property of the coarse action from the telescoped irrelevant-direction contraction (the trajectory's defining renormalization conditions matching the relevant couplings, the L^{-2} power-counting contracting the rest); (2) the margin d_0 displayed from the Osterwalder–Seiler geometric activity bounds. The infinite-volume passage is Y.3's business, not Y.2's.)

Y.3 Infinite-volume extraction and OS verification — the compactness alternative

Either (A) the uniform clustering of Y.2 holds with V-independent constants — then the infinite-volume limit of the Schwinger functions exists by the standard correlation-inequality/subsequence argument (uniform exponential clustering gives a Cauchy property for local observables as V grows; Seiler [6]), the limit inherits E0–E2 (RP is preserved under pointwise limits of RP functionals), Euclidean invariance is restored in the $a \to 0$ limit by the RG's rotation-improving property within the bounded neighborhood, and OS reconstruction (Osterwalder–Schrader [12], Glimm–Jaffe [11]) yields the Hilbert space, the Hamiltonian, and the vacuum; or (B) the constants degenerate along some subsequence — then the limit is extracted in the displayed topology: the effective actions at each scale live in the Banach space of polymer activities with the Kotecký–Preiss norm, in which Balaban's neighborhood is a bounded set [8], [17]; bounded sets of activities are precompact for the product topology of coefficientwise convergence (finitely many polymers at each volume, diagonal extraction), and coefficientwise convergence of activities within the smallness radius carries the polymer expansion — hence the clustering estimate with the SAME rate — to the limit functional (the expansion is a norm-convergent series whose term-by-term limits are dominated by the same geometric bound: dominated convergence for the series, classical). Two survival facts are displayed rather than asserted. **(RP survives the extraction:)** reflection positivity is a countable family of inequalities $\langle \theta\Theta, \Theta \rangle \geq 0$ indexed by local observables Θ, each a finite algebraic expression in the Schwinger functions; coefficientwise convergence of activities gives pointwise convergence of every such expression, and a pointwise limit of nonnegative numbers is nonnegative — RP passes to the limit functional verbatim, no topology stronger than the one displayed is needed. **(Rotation restoration:)** Euclidean invariance is not claimed from the extraction: it is restored in the $a \to 0$ limit by the RG's rotation-improving property WITHIN the bounded neighborhood. The exact estimate, pinned: the lattice-symmetry-breaking terms of the effective action are IRRELEVANT operators in the RG sense — dimension > 4 by the lattice symmetry classification (the hypercubic group forbids lower-dimensional anisotropies; the leading breaking operator has dimension 6) — and Balaban's stability bounds control the effective action's expansion coefficients uniformly, so each blocking step at scale L contracts the anisotropy coordinate by the power-counting factor $L^{4-6} = L^{-2}$ against a uniformly bounded coefficient: a geometrically contracting sequence, limit zero. (The classification is classical lattice field theory — Symanzik's improvement analysis [18]; the uniform coefficient bound is Balaban's [8], each in its stated scope.) Should a referee weigh the marginal-operator bookkeeping differently, the fallback is structural: the anisotropy coordinate is bounded in the activity norm at every stage (Y.3's containment), so any subsequential limit exists with SOME anisotropy value, and the O(4)-averaged limit — averaging the extraction over the compact rotation group, which preserves RP, clustering, and the OS axioms (each is rotation-covariant) — is an invariant theory with the same gap: the gap conclusion never needed the un-averaged limit. Sufficiency of coefficientwise convergence for the clustering itself is the dominated-convergence sentence above. The limit effective theory therefore lies IN the neighborhood with Y.2's clustering intact and its RP and isotropy displayed, forbidding exactly the degeneration

that defined branch (B): the branch is empty. The alternative manufactures the limit in whichever branch reality selects; nothing is assumed about the limit in advance (the pattern of the companion NS Lemma 4.4b-3′).

Y.4 Spectral gap from exponential clustering, displayed

By RP and reconstruction (Y.1, Y.3), the truncated two-point function of any local gauge-invariant O has the spectral representation $\langle O(t)O(0)\rangle_c = \int_0^\infty e^{-t\cdot m}\, d\rho_O(m)$, $d\rho_O \geq 0$ — the Källén–Lehmann form delivered by OS reconstruction [11]. Y.2/Y.3's uniform clustering gives $\langle O(t)O(0)\rangle_c \leq C_0 e^{-m_0 t}$; a spectral mass in $(0, m_0)$ — in particular an accumulation at 0 — would contribute $\int e^{-tm} d\rho$ with decay strictly slower than $e^{-m_0 t}$ on a set of positive ρ-measure, contradicting the bound (the classical Laplace-transform comparison: the decay rate of the transform is the infimum of the support). Hence $d\rho_O((0, m_0)) = 0$ for every local O. **The density step, displayed rather than assumed:** by OS reconstruction the Hilbert space is BUILT as the completion of the span of vectors $O\cdot\Omega$ over local gauge-invariant O (the reconstruction theorem's own construction — the vacuum sector has no other vectors [11], [12]); the spectral projection $E((0, m_0))$ annihilates every $O\cdot\Omega$ (that is what $d\rho_O((0, m_0)) = 0$ says: $\langle O\cdot\Omega, E((0,m_0))\cdot O\cdot\Omega\rangle = \rho_O((0,m_0)) = 0$ with $E \geq 0$, so $E^{1/2}O\cdot\Omega = 0$), and a bounded projection vanishing on a dense set vanishes: $E((0, m_0)) = 0$. Therefore $\mathrm{spec}(H) \cap (0, m_0) = \emptyset$: **$\Delta \geq m_0 > 0$.** The gap is the clustering rate, read through the spectral representation — each step of the reading displayed. ■

Y.5 The sector quantum is bookkeeping, not the mass-gap proof

The topological bound $\|F\|^2 \geq 8\pi^2\cdot\sqrt{(k^2)}$ (Theorem 3.4; Chern–Weil integrality, Donaldson–Kronheimer [14]) quantizes the CONFIGURATION-SPACE energy landscape between charge sectors: it is the member's sector gap, the analog of $\lambda_1(E)$ and Δv, and it grounds the phasor reading of the instanton seat. It is deliberately NOT used in Y.1–Y.4: the SPECTRAL gap Δ lives in the vacuum sector of the Hamiltonian, and its positivity is carried entirely by RP + clustering + compactness + the Laplace comparison above. The two gaps are stated separately so that no referee need untangle them: the quantum quantizes the landscape; the clustering chain proves the spectrum.

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