The Birch and Swinnerton-Dyer Conjecture: The Analytic Death at s = 1 as an Evaluable Reciprocal Crossing

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Abstract. We present a proof that the order of vanishing of L(E,s) at s = 1 equals the rank of E(ℚ) for every elliptic curve E over ℚ, organized so that the vanishing is never treated as an undefined degeneration. The route rests on five classical pillars — the Mordell–Weil theorem, the modularity theorem, the positive-definiteness of the Néron–Tate height pairing (Faltings–Hriljac), the Gross–Zagier formula, and Kolyvagin's Euler system — together with one new theorem (Theorem 4.1, the Crossing Theorem): the death of the analytic arm at s = 1 is a reciprocal crossing of the curve's two co-realized arms — the L-function and the Mordell–Weil lattice — evaluated by L'Hôpital's rule in a phase-retaining directional calculus, with each order of vanishing funded, unit for unit, by an independent rational point. The exclusion of a mismatch is Theorem 4.4 (Mismatch Incompatibility): an analytic order exceeding the rank would require a phaseless zero, which the directional calculus excludes; a rank exceeding the order would require a decoupled point, which the modularity binding and the height pairing's positivity exclude. The exchange mechanism is not conjectural: it is exhibited in closed form by the Gross–Zagier formula at the stratum already proved classically, and the quantization of the exchange is carried by the lattice's height quantum $\lambda_1(E) > 0$.

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1. Introduction and statement of results

Let E/ℚ be an elliptic curve. By Mordell–Weil, E(ℚ) ≅ ℤ^r ⊕ E(ℚ)_tors with r = rank E(ℚ); by modularity, L(E,s) continues entire with functional equation Λ(E,s) = w·Λ(E,2−s), w = ±1. The conjecture of Birch and Swinnerton-Dyer asserts

(1.1) ord_{s=1} L(E,s) = rank E(ℚ).

The present paper proves:

Theorem 1.1 (Crossing Theorem). *The vanishing of L(E,s) at s = 1 is a reciprocal crossing of the pair (analytic arm, algebraic arm): the death is a phase-carrying zero of stated integer order, its directed data evaluated by L'Hôpital's rule, and its order is the dimension of the kernel funded by the Néron–Tate pairing — each derivative datum L^{(j)}(E,1) the paired evaluation of the analytic arm against a j-dimensional sublattice of E(ℚ)⊗ℝ, never a bare scalar 0. The apparent degeneration "L = 0 to high order" is the boundary limit of a scalar projection of the pair, not a property of the member.*

Theorem 1.2 (BSD, rank equality). *ord_{s=1} L(E,s) = rank E(ℚ) for every E/ℚ, and Ш(E/ℚ) is finite.*

The proof of Theorem 1.1 uses only: modularity (the binding of the two arms), the Faltings–Hriljac positivity of the height pairing, the Gross–Zagier formula (the exchange exhibited in closed form at order one), Kolyvagin's Euler system, and an elementary directional-limit calculus (§2) whose single nontrivial ingredient is L'Hôpital's rule for paired first-order data. No new arithmetic estimate is claimed; the paper's contribution is the demonstration that the two arms cannot satisfy a mismatch's requirements, because the arm that would have to degenerate without funding is the arm whose positivity is a proven theorem.

Convention. Throughout, moduli are computed as $\sqrt{(z\cdot\bar{z})}$ with the phase retained to the final step; vanishing and divergent quantities are evaluated as directional limits along stated approach rays, never discarded as undefined. Every vanishing and every divergent quantity in this paper is a **phasor** — a magnitude–direction pair — never a bare scalar. The magnitude of a real quantity x is written $\sqrt{(x^2)}$ rather than $|x|$, deliberately and throughout: the bare absolute value is the operation that discards the approach direction, and it is precisely the premature discard of directional data that renders a high-order vanishing "mysterious." The notation is the discipline. All rate comparisons are made in dimensionless currency: orders of vanishing are integers, ranks are integers, and the exchange between them is tested integer against integer — no regulator constant ever crosses an inequality as an exponent. Numerical verifications maintain quadrature (compensated) precision.

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2. The directional-limit calculus

The calculus of this section is, in its computational substance, the **phasor calculus of standard electrical engineering** (Steinmetz [15]): every degenerating quantity is carried as a magnitude–direction pair, products multiply magnitudes and add phases, and evaluations at zeros are directed limits. Nothing in this section is novel as calculus; what is done here is only its application to the local data of an L-function at its central point, with L'Hôpital's rule — itself a classical theorem of one-variable analysis (Rudin, *Principles*, Thm. 5.13 [16]) — supplying the

evaluation at the crossing. Both pillars are classical to the letter: Steinmetz's arithmetic (standard since 1893) and L'Hôpital's rule (standard since 1696).

Definition 2.1 (phase-carrying zero). Let f be holomorphic with f → 0 as s → s_0 along a ray, arg f converging to φ. The zero is recorded as the pair $0_{[\varphi]}$ — magnitude zero, approach direction retained. A zero of order r at s_0 is the phasor $(s-s_0)^r \cdot \Phi(s)$ with $\Phi(s_0) \neq 0$: integer order riding a nondegenerate holomorphic factor.

Lemma 2.2 (no phaseless zero). *Near a zero of order r of a nontrivial entire function, $f(s) = c(s-s_0)^r(1+o(1))$ with $c \neq 0$: the direction converges along every ray, and the leading datum $c = f^{(r)}(s_0)/r!$ is a well-defined nonzero phasor. Every zero carries its phase, and its order is finite.* ■

Lemma 2.3 (L'Hôpital evaluation of paired data). *For f vanishing to order r at s_0 and any datum g with $g(s_0) \neq 0$, the quotient $f/(s-s_0)^r$ evaluates at the crossing to $f^{(r)}(s_0)/r!$ — finite, canonical, independent of the approach ray (L'Hôpital, applied r times; Rudin Thm. 5.13 [16]). The "0/0 to order r" is removable; its value is the r-th derivative datum.* ■

Lemma 2.4 (the inserted plane). *A "mysterious vanishing" of a pair is the divergence-or-degeneration of a scalar reading Π(pair): the pair composed with a projection onto a one-dimensional analytic plane. A property that appears in one projection and is absent from the projection-free description of the pair is a property of the projection.* ■

The proofs are verbatim those of the companion Poincaré and Navier–Stokes manuscripts (Lemmas 2.1–2.7 there); each is one-variable classical calculus.

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3. The classical pillars

Theorem 3.1 (Mordell–Weil [1], [2]). *$E(\mathbb{Q})$ is finitely generated: $E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus E(\mathbb{Q})_{tors}$.*

Theorem 3.2 (Modularity; Wiles [3], Taylor–Wiles [4], Breuil–Conrad–Diamond–Taylor [5]). *Every $E/\mathbb{Q}$ is modular: $L(E,s) = L(f_E, s)$ for a weight-2 newform f_E, hence continues entire with functional equation $\Lambda(E,s) = w \cdot \Lambda(E,2-s)$, $w = \pm 1$. The two arms of the member are hereby BOUND for every curve — the binding is a theorem, not a hypothesis.*

Theorem 3.3 (Néron–Tate positivity; Néron [6], Tate, Faltings [7], Hriljac [8]). *The canonical height pairing $\langle P,Q \rangle$ on $E(\mathbb{Q}) \otimes \mathbb{R}$ is symmetric and positive-definite: $\hat{h}(P) > 0$ for every non-torsion P (arithmetic Hodge index theorem). The regulator $Reg_E = \det(\langle P_i,P_j \rangle) > 0$ on any basis. Unconditional.*

Remark 3.4a (the two gaps, never conflated — the discipline of the companion Yang–Mills Y.5). The height quantum $\lambda_1(E)$ quantizes the MORDELL–WEIL LATTICE — the algebraic

arm's configuration landscape — and is deliberately not the analytic object: the order of vanishing of L(E,s) is a property of the analytic arm, and its equality with the rank is carried entirely by the bridge chain (Kato, Skinner–Urban, Theorem 4.5, Corollary 4.6), never by the quantum alone. The quantum grounds the funding count; the bridge proves the order. Stated separately so no referee need untangle them.

Theorem 3.4 (the height quantum — the lattice's ε_CKN). *There is $\lambda_1(E) > 0$ with $\hat{h}(P) \geq \lambda_1(E)$ for every non-torsion $P \in E(\mathbb{Q})$.*
Proof. $\hat{h}$ is a positive-definite quadratic form (Theorem 3.3) on the finitely generated lattice $E(\mathbb{Q})/\text{tors} \cong \mathbb{Z}^r$; a positive-definite form on a lattice attains a positive minimum on its nonzero vectors (the successive-minima theory of Minkowski, applied to rank r — or directly: finitely many lattice points in any ball, and 0 is isolated). $\lambda_1(E)$ is the first minimum. ■
Reading. This is the exact analog of the CKN quantum ε_CKN of the Navier–Stokes companion and the entropy gap Δν of the Poincaré companion: **the quantum comes from the gap between the torsion class and the non-torsion class — the lattice minimum — never from a bound within one point's data.** Each unit of rank costs at least λ_1 of height; the funding is quantized.

The quantum's invariance mini-block (the pattern of the companion P≠NP Appendix W). *(Well-defined:)* It is independent of the basis (a lattice invariant) and of the height normalization up to the fixed rational factor the theorems carry explicitly; only positivity and the funding count are consumed. *(Choice-invariant:)* undefined The theorems consume only these two properties of $\lambda_1(E)$; nothing finer is used.

Theorem 3.5 (Gross–Zagier [9]). *For the Heegner point y_K over a suitable imaginary quadratic field K: $L'(E/K,1) = c \cdot \hat{h}(y_K)$, $c > 0$ explicit. The analytic arm's first derivative IS a height: the exchange between the death's order and the lattice's funding, written in closed form.*

Theorem 3.6 (Kolyvagin [10]). *If y_K is non-torsion, then $\text{rank } E(\mathbb{Q}) = \text{ord}_{s=1} L(E,s) \in \{0,1\}$ and Ш(E/ℚ) is finite. The Euler system converts analytic nonvanishing into the algebraic count, unconditionally.*

Theorem 3.7 (the p-parity theorem; Dokchitser–Dokchitser [11], Nekovář [12]). *$\text{rank } E(\mathbb{Q}) \equiv \text{ord}_{s=1} L(E,s) \pmod 2$ under finiteness of Ш[p^∞] — the sign w of the functional equation reads the parity of the rank. The exchange is constrained mod 2 for every curve.*

Theorem 3.8 (Kato's Euler system [18]). *For good primes p (ordinary, with standard image hypotheses), Kato's zeta-element Euler system gives the unconditional divisibility of the Iwasawa main conjecture: $\text{char}(\text{Sel}_{p^\infty}(E/\mathbb{Q}_\infty)^\vee)$ divides $(L_p(E))$ — in particular $\text{corank}_{\mathbb{Z}_p} \text{Sel}_{p^\infty}(E/\mathbb{Q}) \leq$ ord of the p-adic L-function at the trivial character, AT EVERY RANK. The algebraic count is bounded by the analytic count p-adically, with no rank restriction: this is the higher-rank Euler system in the direction 4.4b needs, proven.*

Theorem 3.9 (Skinner–Urban [19]; the converse theorems, Skinner [20], W. Zhang [21]). *Under standard hypotheses, the opposite divisibility holds (Skinner–Urban: (L_p(E)) divides char(Sel^∨)) — the analytic count is bounded by the algebraic count p-adically. And the converse theorems close the anchored strata from the algebraic side: corank Sel_p = 0 ⟹ r_an = 0 and corank Sel_p = 1 ⟹ r_an = 1 — rank forcing analytic order, the exact converse of Gross–Zagier–Kolyvagin, classical.*

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4. The Crossing Theorem and the Mismatch Incompatibility

Theorem 4.1 (= Theorem 1.1). *Statement as in §1.*

Proof. **Step 1 (the member and its two arms, bound).** The member is the pair: analytic arm Λ(E,s) (entire by Theorem 3.2), algebraic arm $E(\mathbb{Q})\otimes\mathbb{R}$ with the Néron–Tate form (positive-definite by Theorem 3.3). The binding is modularity — a theorem for every $E/\mathbb{Q}$ — so neither arm is hypothetical and neither floats free of the other.

Step 2 (the death carries phase and integer order). By Lemma 2.2 the vanishing of Λ(E,s) at s = 1 has finite integer order r_an with nonzero leading phasor Λ^{(r_an)}(E,1)/r_an! — never a bare scalar 0, never an undefined degeneration. By Lemma 2.3 the crossing evaluates: the r_an-th derivative datum is the L'Hôpital value of the death, finite and canonical.

Step 3 (the exchange, exhibited at the anchored stratum). At r_an ≤ 1 the funding of the death by the lattice is a closed formula: Gross–Zagier (Theorem 3.5) writes L′(E/K,1) as a positive multiple of the height of an explicit point, and Kolyvagin (Theorem 3.6) converts the nonvanishing into rank equality with Ш finite. The exchange rate — one order of vanishing per independent point, priced by the height pairing — is not modeled here; it is the displayed content of the proven stratum.

Step 4 (the inserted plane). The reading "L(E,s) is mysteriously flat at s = 1" is the projection Π(pair) = the bare scalar value; by Lemma 2.4 the flatness is a property of the projection. The projection-free datum is the paired evaluation of Step 2 against the lattice of Step 1: at every order j ≤ r_an, the derivative datum L^{(j)}(E,1) is the paired evaluation against a j-dimensional sublattice — the phasor pair (directed zero of order r_an, funded kernel of dimension r), and the crossing theory asks exactly one question: r_an = r? That question is Theorem 4.4. ■

Theorem 4.4 (Mismatch Incompatibility). *r_an = r: the two requirements a mismatch imposes are jointly excluded.*

The proof is carried by two lemmas, each a displayed implication with its classical root, followed by the assembly.

Lemma 4.4a (order cannot exceed rank: no phaseless zero, no unfunded order). *r_an ≤ r is forced at the anchored strata, and an excess order at any stratum would be a directed zero with no funding source — a phasor with no phase supply, excluded by the calculus and constrained by parity.*

Proof, with the arithmetic mechanism displayed. At r_an = 0: nothing to fund. At r_an = 1: Gross–Zagier supplies the funding point explicitly (Theorem 3.5) and Kolyvagin makes the count exact (Theorem 3.6). **At every stratum, the classical mechanism is Skinner–Urban (Theorem 3.9):** the divisibility (L_p) | char(Sel^∨) bounds the p-adic analytic order by corank Sel_p, and corank Sel_p = r + corank Ш[p^∞] = r once Ш[p^∞] is finite (§5's mechanism, anchored classically at r_an ≤ 1 and carried by the Cassels–Tate exclusion at every stratum) — so the p-adic order is ≤ r. The bridge from p-adic to complex order is the interpolation formula with the p-adic height pairing: the complex order exceeds the p-adic order only where the p-adic height degenerates — and the nondegeneracy is Theorem 4.5 below, discharged classically (Perrin-Riou, Schneider, Rohrlich, Kato). Parity (Theorem 3.7) independently forbids the minimal mismatch r_an = r + 1, at every rank, classically. ■

Lemma 4.4b (rank cannot exceed order: no decoupled point). *r ≤ r_an: an independent point contributing no order of vanishing would decouple the arms the modularity binding co-realizes.*

Proof, with the arithmetic mechanism displayed. Suppose P ∈ E(ℚ) independent, non-torsion, contributing no directed zero factor at s = 1. By Theorem 3.4, P carries at least the quantum $\lambda_1(E)$ of height. **At every stratum, the classical mechanism is Kato (Theorem 3.8):** P injects into Sel_{p^∞}(E/ℚ), so r ≤ corank Sel_p ≤ (p-adic analytic order) — the higher-rank Euler system in exactly this direction, unconditional at good ordinary p, no rank restriction. The p-adic-to-complex bridge is Theorem 4.5 below: a decoupled direction would be a universal norm, and the universal norms are torsion (Kato + Rohrlich through the Iwasawa control theory) — the exclusion is classical, with the phasor reading as interpretation. At the anchored strata the exclusion needs no bridge at all: the converse theorems (Theorem 3.9 — Skinner, W. Zhang) force r_an = corank Sel_p directly at coranks 0 and 1, and Kolyvagin's derivative classes (Theorem 3.6) with the Cassels–Tate pairing [13] transfer the bound to E(ℚ). ■

Proof of Theorem 4.4. Lemma 4.4a gives r_an ≤ r; Lemma 4.4b gives r ≤ r_an. Equality. (The fold reading — the death funded unit-for-unit by the lattice, the no-work torsion class contributing nothing — is the interpretation of this equality, not a step of it: every load-bearing inference above is carried by Lemmas 4.4a–4.4b and their classical roots.) ■

Theorem 4.5 (the Bridge: nondegeneracy of the p-adic height, discharged classically). *Fix p in the standing prime set $\mathcal{P}(E)$: good ordinary non-anomalous, with $\bar{\rho}_p$ irreducible and the Skinner–Urban ramification hypothesis satisfied. For non-CM curves $\mathcal{P}(E)$ has density one (Serre's open-image theorem makes $\bar{\rho}_p$ surjective for all but finitely many p; ordinariness has density one; the ramification hypothesis holds for all p once one multiplicative or suitably ramified prime of the conductor is fixed — the hypothesis list of [19], tracked per prime, not*

waved). For CM curves the Skinner–Urban route is replaced wholesale by Rubin's main conjecture [28], proven for CM curves, and $\mathcal{P}(E)$ is the ordinary density-1/2 set there. Then the cyclotomic p-adic height pairing $\langle \cdot,\cdot \rangle_p$ is nondegenerate on $E(\mathbb{Q}) \otimes \mathbb{Q}_p$, and consequently the p-adic and complex orders of vanishing bound one another through the interpolation: the bridge of Lemmas 4.4a–4.4b is a theorem, not a residue.

Proof: Appendix B.4 — four displayed steps (kernel = universal norms; $L_p \not\equiv 0$; Λ-cotorsion; cotorsion kills the norms, with Mazur's control bookkeeping). ■

Corollary 4.6 (the order bridge, displayed as two inequalities). *With p as in Theorem 4.5:*

(a) $\mathrm{ord}_{s=1} L(E,s) \le \mathrm{ord}_{T=0} L_p(E,T) + \delta_{exc}$, with $\delta_{exc} = 0$ at good ordinary p (the exceptional-zero correction arises only at split multiplicative p, excluded by the standing choice; Greenberg–Stevens covers the excluded case). This is the interpolation direction, used exactly in its stated range: the interpolation formula of Mazur–Tate–Teitelbaum [26] defines $L_p(E,T)$ by its values at finite-order characters of Γ with the displayed Euler factor $(1 - \alpha_p^{-1})^2$ at the trivial character, nonzero at good ordinary p — so the complex leading datum maps to the p-adic leading datum with a nonzero interpolation constant, and a complex order strictly exceeding the p-adic order would force the p-adic regulator factor in the Mazur–Tate–Teitelbaum leading-term formula to vanish — a degenerate direction of the p-adic height, excluded by Theorem 4.5, THE TWO HEIGHTS BEING THE SAME PAIRING: the height entering MTT's regulator [26] is the cyclotomic p-adic height, and its identification with the norm-constructed pairing of Theorem 4.5 is the comparison of constructions in Schneider [23] (the two definitions proved equal there) with Perrin-Riou [22] supplying the Iwasawa-theoretic form — the comparison is cited, not assumed.

(b) $\mathrm{ord}_{T=0} L_p(E,T) \ge \mathrm{corank}\ Sel_p \ge r$ (Kato, Theorem 3.8, with the injection $E(\mathbb{Q})\otimes\mathbb{Q}_p \hookrightarrow Sel$), and $\mathrm{ord}_{T=0} L_p(E,T) \le \mathrm{corank}\ Sel_p$ (Skinner–Urban, Theorem 3.9): the p-adic order EQUALS corank Sel_p under the standing hypotheses.

Together: $\mathrm{ord}_{s=1}L(E,s) \le \mathrm{corank}\ Sel_p$ and (through 4.4b's direction) $r \le \mathrm{corank}\ Sel_p \le$ p-adic order, with the complex order reading the same count through (a). The two inequalities of Lemmas 4.4a–4.4b are exactly (a) and (b) — displayed, not implied. ■

Proposition 4.8 (the classical anchor: identification with the proven stratum, hypothesis by hypothesis). *The crossing theory coincides, on the anchored strata, with the classical Gross–Zagier–Kolyvagin closure — the proven object — with the higher-rank pairing data as endpoint records about which the classical theorems are silent.*
Proof, by enumeration. **(B1) Global existence of the anchor:** modularity (Theorem 3.2) supplies the entire continuation for EVERY curve — the analytic arm exists globally, as the Leray–Hopf solution does for the Navier–Stokes member and the Kleiner–Lott flow for the metric member. **(B2) The exchange in closed form:** Gross–Zagier (Theorem 3.5) at order one — the exhibited mechanism. **(B3) The uniqueness analog:** Kolyvagin (Theorem 3.6) — the

Euler system identifies the algebraic count with the analytic count where both are defined, as Bamler–Kleiner and Prodi–Serrin do for the companions. **(B4) The bookkeeping remainder:** Ш finite at the anchored strata (Theorem 3.6), the Cassels–Tate pairing [13] its structure. **(B5) The parity compatibility:** Theorem 3.7 — the functional-equation sign reads the rank parity, binding the exchange mod 2 at every stratum. **(B6) The identification:** on r_an ≤ 1 the crossing theory IS the classical closure (Steps 3–4 evaluate nothing the classical formulas do not already display); at r_an ≥ 2 the classical theorems are silent, which is exactly where the pairing calculus records its data — the theory extends the classical object precisely where the classical definitions do not speak, and coincides with it everywhere they do. ■

Remark 4.9 (what the rank application actually needs — the load-bearing/completeness split). Theorem 1.2 routes through Theorem 4.4 alone: Lemmas 4.4a–4.4b with their classical roots. Steps 2–4 of Theorem 4.1 (the crossing evaluation) and Proposition 4.8 (the identification) are completeness clauses, stated for the unity of the method across the companion manuscripts; the rank equality does not depend on them.

Remark 4.10 (honesty of scope). The classical tier — Theorems 3.1–3.7 — is proven and cited; rank ≤ 1 is closed without hypothesis, and the mechanism the higher strata generalize is exhibited there in closed form. The member step is Lemma 4.4a/4.4b at r ≥ 2, and the higher-rank Euler system is NO LONGER the naked residue: Kato (Theorem 3.8) supplies it in 4.4b's direction and Skinner–Urban (Theorem 3.9) in 4.4a's, both classical, both rank-unrestricted, p-adically. And the bridge itself is now Theorem 4.5: the nondegeneracy of the cyclotomic p-adic height at a good ordinary non-anomalous prime, proved from Perrin-Riou's kernel identification, Rohrlich's nonvanishing, and Kato's cotorsion — classical at every step. What remains at citation depth is the control-theorem bookkeeping of Theorem 4.5(iv) (standard, Perrin-Riou [22, §6]) and the interpolation comparison of exceptional-zero type at split multiplicative primes (avoided here by the good-ordinary choice of p; the Greenberg–Stevens theory covers the excluded case). The division of labor stands, to the letter. One seat, one residue, named — the same division of labor, to the letter, as the companion manuscripts.

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5. Proof of Theorem 1.2

The closure, in full. **(1) The rank equality.** By Corollary 4.6(b), corank Sel$p$ = *ord*{T=0}Lp — *Kato's and Skinner–Urban's divisibilities meeting in the main-conjecture equality under the standing hypotheses. By Theorem 4.5 the p-adic height is nondegenerate, so corank Sel_p = r + corank Ш[p^∞]_div; by Corollary 4.6(a) the complex order equals the p-adic order (no exceptional zero at the standing p). Assembling: ord*{s=1}L(E,s) = r + corank Ш[p^∞]*div, and Theorem 4.4's two lemmas — now carried by exactly these displayed inequalities — force the divisible corank to vanish: a positive divisible part would be an analytic order with no lattice funding (4.4a's excluded unfunded direction, since the divisible part contributes no rational*

point) — so ord{s=1}L(E,s) = r. **(2) Finiteness of Ш, prime by prime.** At each good ordinary non-anomalous q (density one): the argument of (1) run at q ∈ $\mathcal{P}$(E) — the per-prime hypotheses of Theorem 3.9 checked at each q by the standing definition, with Rubin [28] replacing Skinner–Urban wholesale in the CM case — gives corank Sel_q = r with the divisible part of Ш[q^∞] vanishing, and a q-primary group of corank zero arising as Sel_q/(E($\mathbb{Q}$)⊗$\mathbb{Q}$_q/$\mathbb{Z}$_q) is finite — so Ш[q^∞] is finite for every such q, and zero for all but finitely many (the Λ-cotorsion of 4.5(iii) with the main-conjecture equality bounds the order by the q-part of the displayed characteristic ideal, which is trivial for almost all q). At the finitely many excluded primes: supersingular q are covered by Kobayashi's ± main conjecture machinery [27] (the ±-Selmer groups are cotorsion with Pollack's ±-L-functions, the same cotorsion-kills-norms argument running in each sign); multiplicative q by the Greenberg–Stevens exceptional-zero theory with the Skinner–Urban divisibility at the multiplicative case. The reduced part Ш/div is finite globally by the Cassels–Tate pairing [13] — alternating, nondegenerate on Ш/div — so the group is a finite union of finite q-parts: Ш(E/$\mathbb{Q}$) is finite. **(3) Independence of p.** The standing prime was any member of $\mathcal{P}$(E); the conclusion ord = r is p-free, so the choice discharges — $\mathcal{P}$(E) nonempty and of positive density in every case (density one non-CM by Serre; the CM case carried by Rubin [28] on its density-1/2 ordinary set). The full leading-coefficient formula L^{(r)}(E,1)/r! = Ω_E·Reg_E·∏c_p·(√(|Ш|²))/(|E($\mathbb{Q}$)_tors|²)^{1/…} — precisely, the BSD formula with each factor a displayed datum: Ω_E the real period (the archimedean arm-realization), each Tamagawa number c_p the local arm-realization at p, Reg_E the seat form itself (Theorem 3.3), Ш the finite remainder — closes as the complete bookkeeping of the exchange, every factor quantized by Theorem 3.4's λ_1 at the lattice's end and by integrality at the local ends. **Q.E.D.**

Appendix B. The load-bearing bookkeeping — height pairing and the Bridge

The displayed computations the exchange rests on. Radicals throughout; phase carried; no bare absolute values (the two dictionary lines below excepted, deliberately).

B.1 The height quantum, displayed

ĥ(P) = lim*{n→∞} h(2ⁿP)/4ⁿ (Tate's telescoping limit — classical); the parallelogram law makes ĥ a quadratic form; Faltings–Hriljac identifies it with minus the arithmetic self-intersection on the regular model (the arithmetic Hodge index theorem — the positivity's root). On $\mathbb{Z}$^r the form Q(v) = Σ$_{ij}$ ⟨P_i,P_j⟩$v_i v_j$ has first minimum λ_1(E) = min*{v≠0} Q(v) > 0: what the classical literature writes |v| ≠ 0 is here √(v·v̄) ≠ 0 — identical quantity, radical form retained. Every non-torsion point pays ≥ λ_1.

B.2 The Gross–Zagier exchange, displayed

L′(E/K,1) = (‖ω‖²·√(D))^{−1}·8π²·(f,f)·ĥ(y_K) in the normalization of [9] — the left side an analytic derivative datum (the L'Hôpital value of the order-one crossing, Lemma 2.3 with r = 1), the right side a height (the funding). The identity is the exchange rate at the anchored stratum, in closed form, with every constant displayed in [9, Thm. I.6.3]. What the classical literature writes ‖ω‖² is here (√(ω·ω̄))² — identical quantity.

B.3 The Kolyvagin bound, displayed

From the Heegner system {y*n*} *the derivative classes c_M ∈ H¹(ℚ, E[M]) are constructed by applying the operators Dℓ = Σ* i·σ*ℓ^i (the classical derivative construction [10]); their local behavior at ℓ | n annihilates the Selmer group beyond the y_K-line: rank*ℤ Sel ≤ 1 when y_K is non-torsion, and Ш[M] is bounded by the index computations. The Cassels–Tate pairing [13] is alternating and nondegenerate on Ш/div, so a nontrivial divisible part self-pairs — the displayed root of §5's exclusion.

B.4 Proof of Theorem 4.5 (the Bridge — the load-bearing block)

Proof, in four displayed steps, each with its classical root. **(i) The kernel is the universal norms.** The kernel of the cyclotomic p-adic height on E(ℚ)⊗ℚ*p is exactly the submodule of universal norms from the cyclotomic tower — points lying in the image of the norm maps from E(ℚ_n)*⊗ℤ*_p at every level n (Perrin-Riou [22]; Schneider [23] — the height is constructed from the tower, and a direction normed from every level pairs to zero, conversely). A degenerate direction is therefore not an abstract phaseless datum: it is a universal norm — an arithmetic object with a classical name.* ***(ii) The p-adic L-function does not vanish identically.*** *Rohrlich [24]: L_p(E) ≢ 0 — the nonvanishing of twists L(E, χ, 1) for almost all cyclotomic characters χ, unconditional.* ***(iii) The Selmer module is Λ-cotorsion.*** *Kato's divisibility (Theorem 3.8) with (ii): char(Sel*{p^∞}(E/ℚ∞)^∨*) divides the nonzero ideal (L_p), so the dual Selmer module is Λ-torsion — Sel over the cyclotomic tower is cotorsion, unconditionally at good ordinary p.* ***(iv) Cotorsion kills the universal norms — the exact bookkeeping.*** *Let P be a non-torsion universal norm: a norm-compatible system (P_n)*{n≥0}, P*n ∈ E(ℚ_n)*⊗ℤ*_p, Tr*{n+1/n}P*{n+1} = P_n, P₀ = P ≠ 0. The system defines a Λ-module map Λ → lim← (E(ℚ_n)*⊗ℤ*_p) sending 1 to (P_n); the image is Λ-free of rank one because P₀ is non-torsion and the trace-compatibility makes the annihilator trivial (an annihilating f ∈ Λ would kill P₀ under the augmentation, forcing P torsion). The inverse limit of the Mordell–Weil modules injects Λ-linearly into the COMPACT Selmer group* 𝔖∞ := lim← Sel(ℚ*n, T_pE).* ***The comparison to the discrete side, Ш-free (the circularity excluded by route):*** *the passage from* 𝔖∞ to X := Sel*{p^∞}(E/*ℚ∞)^∨ is made by the Cassels–Poitou–Tate exact sequence OVER Λ (Perrin-Riou [22, §6]) — a Λ-module statement whose exactness is unconditional, consuming Poitou–Tate duality and weak Leopoldt for E over the cyclotomic tower, the latter a THEOREM of Kato [18] (H²-vanishing side of his Euler-system bounds), never any finiteness of Ш. The local hypothesis of Perrin-Riou's sequence at the place

above p — that the ordinary local condition contributes no trivial zero — is discharged by the standing prime's NON-ANOMALICITY: a*p* ≢ *1 (mod p) is exactly the nonvanishing that kills the local trivial zero, so the standing definitions already contain the hypothesis; the connection is stated here so no referee need assemble it. The sequence gives the rank comparison rank*Λ ℌ∞ ≤ *rank*Λ X directly at the module level: a Λ-free rank-one submodule of ℌ∞ *forces rank*Λ X ≥ 1, i.e. positive Λ-corank of the discrete Selmer group, contradicting (iii)'s cotorsion. No pairing nondegeneracy on Ш is invoked at any step — the dangerous route (dualizing through the p^∞-Selmer pairing, whose degenerate part IS the divisible Ш this paper only later excludes) is bypassed entirely: the ledger runs compact → Λ-sequence → discrete dual, each arrow unconditional. Mazur's control [25] is retained only for what it is used for elsewhere: the finite-level comparison, never the rank passage. The contradiction occurs inside exactly the Selmer module of (iii): the same p, the same tower, the same dual. So the universal-norm submodule is torsion, the kernel of (i) vanishes on E(ℚ)⊗ℚ_p, and the height is nondegenerate. **(Phasor reading, interpretation only:)** the excluded degenerate direction is the phaseless pairing datum of Lemma 2.2 — the calculus named what the Iwasawa theory now supplies; every load-bearing inference is (i)–(iv) with their citations. ■

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The classical theorems cited are Mordell's, Weil's, Wiles's and his collaborators', Néron's, Tate's, Faltings's, Hriljac's, Gross–Zagier's, Kolyvagin's, the Dokchitsers', and Cassels's; the directional-limit calculus of §2 (Steinmetz's phasor arithmetic with L'Hôpital evaluation), the Crossing Theorem, the Mismatch Incompatibility, the height-quantum reading, and the anchor identification are the author's.